$$\left(\sum \phi_n z^n\right)^d = \phi_0^{d\cdot} + d\cdot\phi_{n_0}\phi_0^{d-1} z^{n_0} + \text{h.o.t.}$$

The coefficient $d\,\phi_{n_0}\phi_0^{d-1} \neq 0$, and the exponent $n_0 < n_0 d$, since $d \geq 2$.
This gives a contradiction (n_0 was minimal). □

Rem: This result works for any field of characteristic 0, as long as $\forall a$, $\exists \alpha$ such that $\alpha^{d-1} = a$. If not, $f \simeq z \mapsto az^d$ in any case.
In characteristic p the situation is far more complicated.
One can see it on the proof of uniqueness for example: $d\phi_{n_0}\phi_0^{d-1} = 0$ if $p \mid d$, $p = \text{char } k$.
A more direct way to notice it is the example: $f(z) = z^p(1+z)$.
The derivative is $f'(z) = pz^{p-1} + (p+1)z^p = z^p$ (in char p.)
while $\tilde{f}(z) = z^p$ has everywhere vanishing derivative. Hence $f \not\simeq_{\text{for}} \tilde{f}$.

Hence, in this case, formal, holomorphic and topological classifications coincide (being $d = \text{ord}_0(f)$ an invariant in all such cases).

III Parabolic / Tangent to the identity case.

We consider now the case of $f:(\mathbb{C},0)\circlearrowleft$ with $\lambda = f'(0)$ a root of unity: $\lambda = e^{2\pi i \frac{p}{q}}$, $q \in \mathbb{N}^*$, $p \in \mathbb{Z}$, $(p,q) = 1$.

In local coordinates, we can write $f(z) = \lambda z(1 + o(1))$.

Notice that the q-th iterate of f will be $f^q(z) = \underset{1}{\underset{\shortparallel}{\lambda^q}} z(1 + o(1))$

Def: $f:(\mathbb{C},0)\circlearrowleft$ is called tangent to the identity if $\lambda = f'(0) = 1$.

In this case, we write $f(z) = z(1 + az^r + o(z^r))$ for some $r \geq 1$, $a \neq 0$

The value $r+1$ is called the multiplicity of f (it is in fact the multiplicity of 0 as a fixed point of f).

Prop: The multiplicity of a tangent to the identity germ is a formal invariant of conjugacy.

Proof: We write $f(z) = z(1+az^r+o(z^r)) = z + az^{r+1} + o(z^{r+1})$, $r \geq 1$

Similarly, we write $\tilde{f}(z) = z(1+bz^s+o(z^s)) = z + bz^{s+1} + o(z^{s+1})$ $s \geq 1$.

Let $\Phi(z) = z\sum_{n\geq 0} \phi_n z^n$ be any formal ~~germ~~ invertible germ $(\phi_0 \neq 0)$.

We want to show that if $\Phi \circ f = \tilde{f} \circ \Phi$, then $r = s$.

We compute $\Phi \circ f$ modulo $(z)^{r+2}$, i.e., we consider expression only up to the coefficients of z^{r+1}.

$$\Phi \circ f(z) \equiv (z + az^{r+1}) \cdot \sum_{j=0}^{r} \phi_j (z + az^{r+1})^j \equiv (z + az^{r+1})\left[\sum_{j=0}^{r} \phi_j z^j\right] \equiv$$

$$\equiv z\sum_{j=0}^{r} \phi_j z^j + \phi_0 a z^{r+1}.$$

Analogously, we compute $\tilde{f} \circ \Phi(z)$ modulo $(z)^{s+2}$:

$$\tilde{f} \circ \Phi(z) \equiv z\sum_{j=0}^{s} \phi_j z^j + b z^{s+1} \phi_0^{s+1}.$$

Notice that if $\Phi \circ f = \tilde{f} \circ \Phi$, then the two expression must coincide also modulo z^{n+1} $\forall n \in \mathbb{N}$

Being $a\phi_0$ and $b\phi_0^{s+1}$ not vanishing, it follows that $r = s$. □

Rem: From the proof of the previous proposition, we may also notice that, up to the linear change of coordinates $z \mapsto \phi_0 z$, with $\phi_0^r = \frac{a}{b}$, we can always conjugate a germ f as above to a germ of the form $\tilde{f}(z) = z + bz^{r+1} + o(z^{r+1})$, for any choice of b. The most common choices are $b=1$ and $b=-1$.

The linear conjugacy is unique up to a r-th root of unity.

Proposition: Let $f:(\mathbb{C},0)\circlearrowleft$ be a (formal) tangent to the identity germ. Then f is formally conjugate to: $\tilde{f}(z) = z(1+z^r+\beta z^{2r})$, $\tilde{=}\, z(1-z^r+\beta z^{2r})$
for some $\beta\in\mathbb{C}$.

The number β is a formal invariant of conjugacy.

~~The conjugacy is uniquely determined up to~~

Proof: $\exists$ We write $f(z)=z(1+a(z))$, where $\mathrm{ord}_0 a = r$,

We expand a in formal power series, and get $1+a(z)=\sum_{j\geq 0} a_j z^j$,
with $a_0=1$, $a_1 \dots a_{r-1}=0$, $a_r=\alpha\neq 0$ ($\alpha=+1$ or -1).

Similarly, we write the candidate normal form as $\tilde{f}(z)=z(1+\tilde{a}(z))$
with $1+\tilde{a}(z)=\sum_{j\geq 0}\tilde{a}_j z^j$, $\tilde{a}_0=1$, $\tilde{a}_1=\dots=\tilde{a}_{r-1}=0$, $\tilde{a}_r=\alpha$.

We consider changes of coordinates of the form $\Phi(z)=z\underbrace{\sum_{j\geq 0}\phi_j z^j}_{1+\phi(z)}$, with $\phi_0=1$.

By computing the expansion in formal power series of $\Phi\circ f$, we get: $\Phi\circ f(z) = z(1+a(z))\cdot\sum_{j\geq 0}\phi_j z^j(1+a(z))^j = z\sum_{j\geq 0}\phi_j z^j(1+a(z))^{j+1} =$

$$= z\cdot\sum_{j\geq 0}\phi_j z^j\cdot\sum_{I\in\mathbb{N}^{j+1}} a_I z^{|I|}$$

where $a_I = a_{i_1}\cdots a_{i_{j+1}}$ and $|I|=i_1+\dots+i_{j+1}$ if $I=(i_1\dots i_{j+1})$. It follows that the n-th coefficient of $\frac{\Phi\circ f(z)}{z}$ is

$$I_n = \sum_{\substack{j\geq 0\\ I\in\mathbb{N}^{j+1}\\ j+|I|=n}}\phi_j a_I.$$

Similarly, for $\tilde{f} \circ \Phi$ we get:

$$\tilde{f} \circ \Phi(z) = z(1+\phi(z)) \cdot \sum_{k \geq 0} \tilde{a}_k z^k \cdot (1+\phi(z))^k = z \sum_{k \geq 0} \tilde{a}_k z^k (1+\phi(z))^{k+1} =$$

$$= z \sum_{k \geq 0} \tilde{a}_k z^k \cdot \sum_{H \in \mathbb{N}^{k+1}} \phi_H z^{|H|}.$$

It follows that the n-th coefficient of $\dfrac{\tilde{f} \circ \Phi(z)}{z}$ is $II_n = \sum_{\substack{k \in \mathbb{N} \\ H \in \mathbb{N}^{k+1} \\ k+|H|=n}} \tilde{a}_k \phi_H$.

The conjugacy equation $\Phi \circ f = \tilde{f} \circ \Phi$ is satisfied if and only if

$I_n = II_n \quad \forall n \in \mathbb{N}$.

We study first $I_n = \sum_{\substack{j \geq 0 \\ I \in \mathbb{N}^{j+1} \\ j+|I|=n}} \phi_j a_I$.

Since $a_i = 0 \ \forall i = 1 \dots r-1$, we have that:

$|I| = 0 \Rightarrow a_I = a_0^{j+1} = 1$, and $|I| < r \Rightarrow a_I = 0$.

If $|I| = r$, then either $a_I = 0$ or $a_I = a_r \cdot a_0^j = \alpha$ (with $j+1$ multi-indices satisfying this condition).

$(n \geq r)$

Hence $I_n = \underbrace{\phi_n}_{j=n} + \underbrace{\phi_{n-r}(n-r+1)\alpha}_{j=n-r} + \mathrm{lot}(\phi_j, j<n-r)$,

a polynomial on ϕ_j, $j<n-r$ and on a_i

We now study $II_n = \sum_{\substack{k \geq 0 \\ H \in \mathbb{N}^{k+1} \\ k+|H|=n}} \tilde{a}_k \phi_H$.

For $k=0$, $|H| = n$, and $\sum_{\substack{|H|=n \\ H \in \mathbb{N}}} \phi_H = \phi_n$.

6.15

For $k=n$, $|H|=0$, and $H=\underbrace{(0 \ldots 0)}_{n+1}$. Hence $\displaystyle\sum_{\substack{|H|=0 \\ H\in\mathbb{N}^{n+1}}} \phi_H = \phi_0^{n+1} = 1$.

For $k=1 \ldots r-1$, $\tilde{a}_k = 0$.

For $k=r$, we have $\displaystyle\sum_{\substack{|H|=n-r \\ H\in\mathbb{N}^{r+1}}} \phi_H = (r+1)\phi_{n-r}\cdot \underset{\substack{\| \\ 1}}{\phi_0^r} + \text{l.o.t}(\phi_h, h<n-r)$.

(1 if $n=r$, already considered for $k=n$) For $k>r$, $\displaystyle\sum_{|H|=n-k} \phi_H = \text{lot}(\phi_h, h<n-r)$.

The equation $\mathbb{E}_n := \mathbb{II}_n - \mathbb{I}_n = 0$ gives:

$$0 = \tilde{a}_n + \cancel{\phi_n} + \alpha\big(\overbrace{(r+1)\,\phi_{n-r}}^{\text{if } n>r}\big) + \text{lot}\,(\phi_j, j<n-r;\ \tilde{a}_k,\ r<k<n)$$
$$- \cancel{\phi_0} - \alpha\big((n-r+1)\,\phi_{n-r}\big) + \text{lot}\,(\phi_j, j<n-r)$$

For $n<r$, the equation $\mathbb{E}_n = 0$ is automatically satisfied.

For $n=r$, we get: $0 = \tilde{a}_r - \alpha\cdot\phi_0$ which is again satisfied since $\tilde{a}_r = \alpha$, $\phi_0 = 1$.

For $n>r$, we get $\tilde{a}_n = \alpha\,\phi_{n-r}\,(n-2r) + \text{lot}\,(\phi_j, j<n-r;\ \tilde{a}_k,\ r<k<n)$. $(*_n)$

If $n = r+1 \ldots 2r-1$, we set $\tilde{a}_n = 0$, and there exists a unique ϕ_{n-r} satisfying $(*_n)$.

If $n=2r$, we set $\phi_r = 0$ (or any other value), and there exists a unique $\tilde{a}_{2r} = \beta$ satisfying $(*_{2r})$

For $n>2r$, again we set $\tilde{a}_n = 0$, and there exists ϕ_{n-r} unique satisfying $(*_n)$.

Note that with $\phi_0 \in \mathbb{C}^*$ general, and $\tilde{a}_r$ to be determined we get:

$$0 = \tilde{a}_n \phi_0^{n+1} + \tilde{a}_r (r+1)\,\phi_{n-r}\,\phi_0^{r\,(n-r)} - \alpha\,(n-2r+1)\,\phi_{n-r} + \text{lot}.$$

For $n=r$, we get $\phi_0^{r+1}\tilde{a}_r = \alpha\phi_0 \to \tilde{a}_r\,\phi_0^r = \alpha$. Using this,

Then the condition $(*_n)$ becomes: $\phi_0^{n+1}\tilde{a}_n = \alpha\phi_{n-r}(n-2r) + \text{lot}$

$\boxed{\beta \text{ is invariant}}$: Suppose $f(z) = z(1+\alpha z^r + \beta' z^{2r})$, $\tilde{f} = z(1+\alpha z^r + \beta z^{2r})$ 6.16

We want to prove that if $\Phi \circ f = \tilde{f} \circ \Phi$, then $\beta = \beta'$.

By the computations done above, $\alpha \phi_0^r = \alpha$, i.e. $\phi_0^r = 1$. (from E_r).

From E_n, $r < n < 2r$, $\tilde{a}_n = 0$, and by unicity of the solution of ϕ_n, we infer $\phi_n = 0 \quad \forall\, 1 < n < r$.

From E_{2r}, we infer $\beta = \tilde{a}_{2r}$, $\phi_0^{2r+1} \beta = \alpha \phi_r (\overset{0}{\overset{\shortparallel}{2r - 2r}}) + \text{lot}(\phi, < r)$

Being $\phi_1 = \dots = \phi_{r-1} = 0$, the contribution from I_{2r} is $\phi_0 \cdot \beta'$, while the contribution from II_{2r} is $\phi_0^{2r+1}\beta$.

We infer $\phi_0^{2r+1}\beta = \phi_0 \beta' \iff \underset{\overset{\shortparallel}{1}}{\phi_0^{2r}}\beta = \beta' \iff \beta = \beta'$. □

Rem: notice that in this case the conjugacy Φ is not uniquely determined: it is determined up to the r-th root of unity ϕ_0, and the free choice of ϕ_r. In other terms, the set of diffeomorphisms commuting with $z(1+\alpha z^r + \beta z^{2r})$ draws r disjoint ~~lines~~ curves in $\widehat{\mathrm{Aut}}(\mathbb{C}, 0)$.

Corollary: For any tangent-to-the-identity germ $f : (\mathbb{C},0) \circlearrowleft$ of multiplicity $r+1$ there exists a holomorphic change of coordinates (in fact, polynomial of degree $\leq 2r$) so that we can write f as $f(z) = z(1 + \alpha z^r + \beta z^{2r} + o(z^{2r}))$ (for any $\alpha \in \mathbb{C}^*$, β is the invariant).

Rem: The invariant β can be computed in a intrinsic way. ~~by the formula~~

First notice that ~~the~~, if we change the α in the normal form, to an $\tilde{\alpha}$, β and $\tilde{\beta}$ are related as follows:

$f(z) = z(1+\alpha z^r + \beta z^{2r})$, $\tilde{f} = z(1+\tilde{\alpha} z^r + \tilde{\beta} z^{2r})$, $\Phi(z) = \phi_0 z$.

$\leadsto \tilde{\tilde{f}}(z) = \Phi^{-1} \circ \tilde{f} \circ \Phi(z) = z(1 + \tilde{\alpha}\phi_0^r z^r + \tilde{\beta}\phi_0^{2r} z^{2r}) \to \alpha = \tilde{\alpha}\phi_0^r \leadsto \phi_0^r = \dfrac{\alpha}{\tilde{\alpha}}$

$\beta = \tilde{\beta}\phi_0^{2r} = \tilde{\beta}\dfrac{\alpha^2}{\tilde{\alpha}^2}$.

Hence the invariant is more properly given by $\frac{\beta}{\alpha^2}$, and called the <u>index</u> of f.

This value can be computed directly on the normal form, as

$$\frac{\beta}{\alpha^2} = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-f(z)}\,dz \;,$$ where γ is any positive (simple) loop around 0.

This expression is also invariant by change of coordinates, since residues are, and any conjugacy map sends any positive loop around 0 to another positive loop around 0.

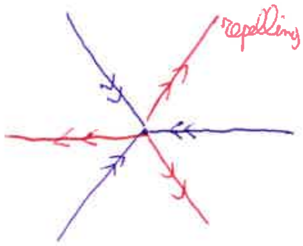
The formal conjugacy does not converge in general, and the holomorphic classification is much more complicated to obtain (or even to state)

To try and understand the behavior of iterates of f around the origin, we start by studying the maps $f(z) = z(1+\alpha z^r)$.

We first notice that if $v \in S^1 = \partial\mathbb{D}$ is a vector so that $\overbrace{\alpha v^r}^{w} > 0$ ← i.e. $\alpha v^r \in \mathbb{R} \cap (0,\infty)$

then $f(t.v) = tv(1+t^r\alpha v^r)$.

For $t>0$, $t(\underset{\mathbb{R}}{1+t^r w}) > t$.



In particular, the half line $\mathbb{R}_+\cdot v$ is invariant by f and orbits are repelled from the origin.

Similarly, if $\alpha v^r < 0$, then ~~$f(tv)$~~ : $t(1+t^r \underset{0}{w}) < t$

notice that $0 < 1 + t^r w < 1 \iff 0 < t < \sqrt[r]{-\frac{1}{w}}$ and the segment $\left[0; \sqrt[r]{-\frac{1}{|\alpha|}}\right]v$

is invariant, and orbits (in the open segment) tend to 0.

Notice that the germ f has exactly r "repelling directions", and r "attracting directions", equally distributed (they are in $2r$-th roots of $\frac{\bar{\alpha}}{\alpha}$)

This leads to the next definition.

Def: Let $f:(\mathbb{C},0)\circlearrowleft$ be a tangent to the identity germ of multiplicity $r+1\geq 2$. $f(z)=z(1+\alpha z^r+o(z^r))$ $(\alpha\neq 0)$

A unit vector $v\in S^1\equiv\partial\mathbb{D}$ is an attracting/repelling direction for f if $\alpha v^r<0$ (resp. $\alpha v^r>0$).

(For $\alpha=-1$, attracting directions are the r-th root of 1, and the repelling direction are the 2r-th roots of unity which are not attracting)

Notice that attracting directions for f are repelling directions for f^{-1} (and vice versa ~~similarly for~~), since $f^{-1}(z)=z(1-\alpha z^r+o(z^r))$.

We want to study the behavior of the orbits that are "close" to attracting (/repelling) directions.

Def: let $f(z)=z(1+\alpha z^r+o(z^r))$ a tangent to the identity germ, and $v\in S^1=\partial\mathbb{D}$ a tangent direction (attracting).

We say that a point z tends to 0 tangent to v if:

$f^n(z)\xrightarrow[n\to\infty]{}0$ and $\dfrac{f^n(z)}{|f^n(z)|}\to v$ (Up to shrinking the domain, we may assume $f^n(z)\neq 0$, see below).

We call the set of points tending to 0 tangently to v the basin of attraction (at 0) centered at v.

Rem: notice that, for local dynamical systems, f is defined only locally, and ~~if~~ the orbit $(f^n(z))_{n\in\mathbb{N}}$ may not be defined for all z in a neighborhood of 0. In fact, to be defined, z must belong to the stable set $K_f(U)=\bigcap_{n\in\mathbb{N}}f^{-n}(U)$

8.13

the collection $(K_f(U))_U$ forms a germ of set K_f, where the dynamics of f is defined.

In the examples we saw; K_f is a neighborhood of 0 when f is attracting or superattracting, while $K_f = \{0\}$ for repelling germs.
For $f(z) = z(1+2z^2)$, K_f contains the attracting segments, but it ~~does not contain~~ is disjoint from the (open) repelling half-lines.

In what follows, we will also need the following definition:

Def: An attracting petal centered at an attracting direction v of f (f tangent to the identity) is a open, simply connected, f-invariant ($f(P) \subseteq P$) set P ($\subseteq K_f \setminus \{0\}$) so that $z \in \mathcal{A}_{0,v}$ the basin of attraction centered at v $\iff$ $O_f(z) \cap P \neq \emptyset$.

A repelling petal is an attracting petal for f^{-1}.

Rem: the definitions of forward / backward / totally invariant sets is slightly different in the local setting than its global counterparts:
a (germ of set) A (so that $0 \in \bar{A}$) is:

- forward invariant if $f(A) \subseteq A$.
- backward invariant if $f^{-1}(A) \subseteq A$. as a germ
- totally (or completely) invariant if $f^{-1}(A) = A$ both forward and backward invariant.

Example: $f: (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, $f(z,w) = (z^2, z(1+w^2))$, $A = \{zw = 0\}$.
$\Rightarrow$ $f^{-1}(A) = \{z=0\} \cup \{z=0\} \cup \{w = \pm i\} \underset{\uparrow}{\simeq} \{z=0\} \subseteq A$.
But $f(A) = \{(z^2, z) \mid z \in \mathbb{C}\} = \{z = w^2\} \not\subseteq A$. as a germ

We can finally state the main result concerning tangent to the identity germs. 6.20

Theorem (Leau-Fatou flower theorem). Let $f:(\mathbb{C},0)\circlearrowleft$ be a tangent to the identity germ of multiplicity $r+1\geq 2$. Let $v_0^+, v_0^-, \dots, v_{r-1}^+, v_{r-1}^-$ be the attracting and repelling directions of f (in cyclic order). Then,

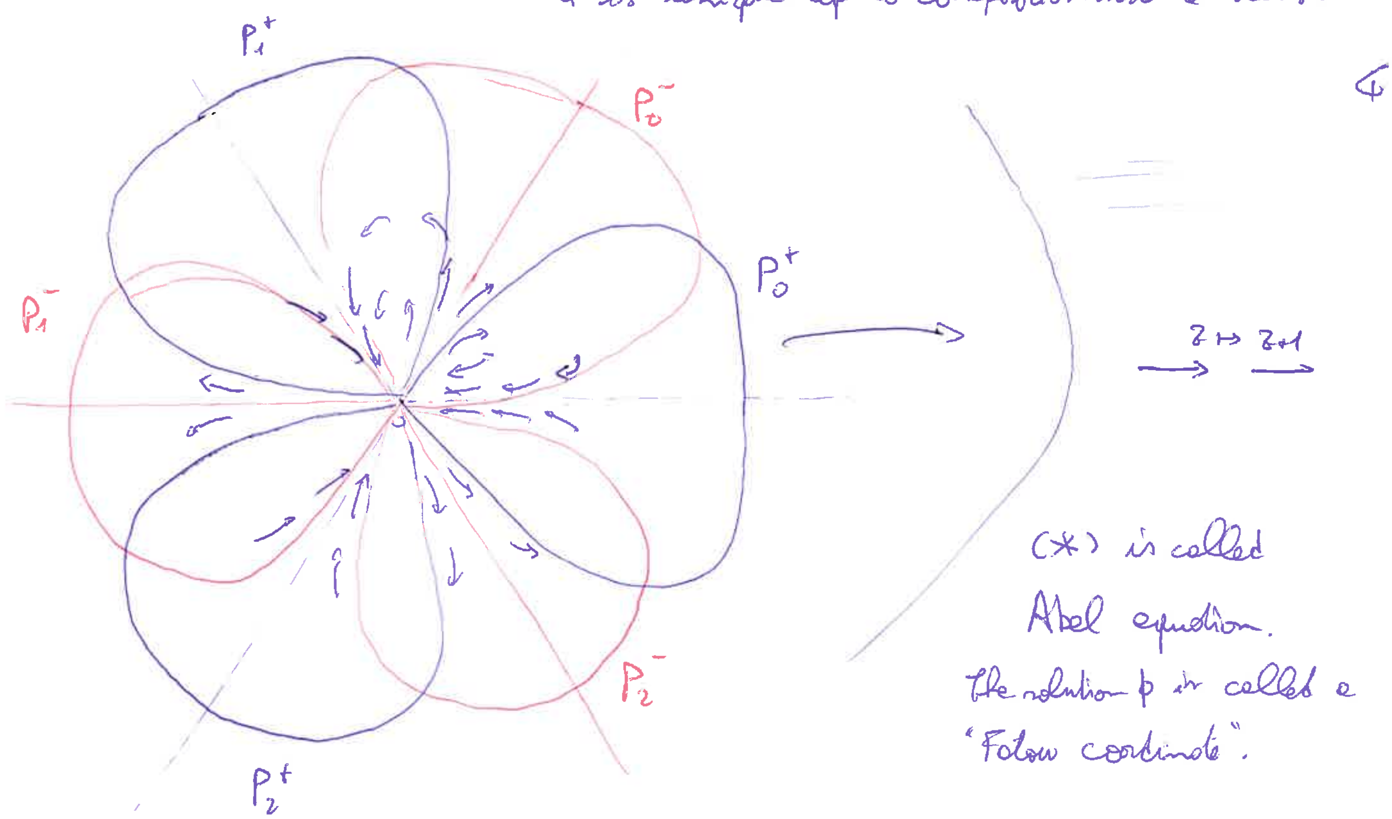
a) for each attracting/repelling direction $v_j^\pm$ there exists an attracting/repelling petal $P_j^\pm$, so that $\bigcup_{j=0}^{r-1} P_j^+ \cup \bigcup_{j=0}^{r-1} P_j^- \cup \{0\}$ is a neighborhood of the origin. Moreover, two petals intersect if and only if they are adjacent (with respect to the cyclic order given on the tangent directions).

- $K_f \setminus \{0\} = \bigcup_{j=0}^{r-1} \mathcal{A}_{0,v_j^+}$, $\mathcal{A}_{0,v_j^+} = \bigcup_{n\in\mathbb{N}} f^{-n}(P_j^+)$ basin of attraction centered at v_j^+.

- $f|_{P_j^+}$ is holomorphically conjugated to the map $z\mapsto z+1$ on the right half plane: i.e. $\exists\, \phi: P_j^+ \to V_j$ biholomorphism $\phi\circ f(z) = \phi(z)+1$ $(*)$, $V\supset\{\operatorname{Re} z > R\}$.

(similarly, $f|_{P_j^-}$ is holomorphically conjugated to $z\mapsto z-1$ on $\{\operatorname{Re} z<-R\}$)

ϕ is unique up to composition with a translation.



$(*)$ is called Abel equation.

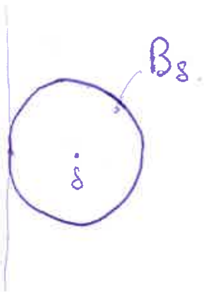
The solution ϕ is called a "Fatou coordinate".

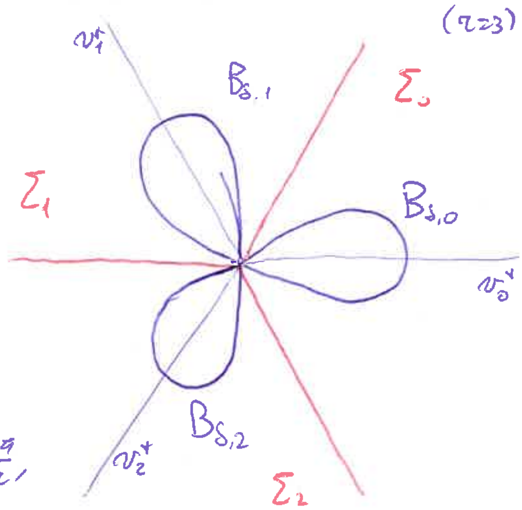
Proof. Write $f(z) = z(1 + a_r z^r + o(z^r))$. Step 1: work locally at ∞. (E.21)

Up to a linear change of coordinates, we may assume $a_r = -1$, so that $v_j^+ = e^{\frac{2\pi i}{r} j}$ are the r-th roots of unity.

$\forall \delta > 0$ let $B_\delta = B(\delta, \delta) = \{ z \in \mathbb{C} \mid |z - \delta| < \delta \}$, the open disc containing 0 in its boundary. Its preimages through $z \mapsto z^r$ has r connected components $B_{\delta,0}, \dots, B_{\delta,r-1}$

We will show that $B_{\delta,0} \dots B_{\delta,r-1}$ are attracting petals (even though too small to give, with the repelling petals, a pointed neighborhood of the origin).





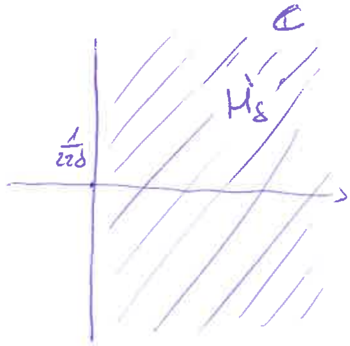
Let Σ_j $j = 0 \dots r-1$ be the sectors centered at v_j^+ of amplitude $\frac{2\pi}{r}$, i.e. $\Sigma_j = \{ z \in \mathbb{C}^* \mid (2j-1)\frac{\pi}{r} < \arg z < (2j+1)\frac{\pi}{r} \}$

Notice that $B_{\delta,j} \subset \Sigma_j$, and it is tangent at 0 to the cone centered at v_j^+ of amplitude $\frac{\pi}{r}$.

We now perform a change of coordinates at Σ_j, to bring 0 to $\infty \in \hat{\mathbb{C}}$:

let $\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $z \mapsto \frac{1}{r z^r}$. This maps sends biholomorphically Σ_j to $\mathbb{C} \setminus \mathbb{R}_-$, with inverse $\psi_j^{-1}(w) = (rw)^{-\frac{1}{r}}$, the r-th root chosen accordingly (so that $\psi_j^{-1}(\frac{1}{r}) = v_j^+$)

Notice that $\psi(B_{\delta,j}) = H'_\delta := \{ w \in \mathbb{C} \mid \operatorname{Re} w > \frac{1}{2r\delta} \} =: H_{\frac{1}{2r\delta}}$



If $|w| \gg 1$ ($\Leftrightarrow |z| \ll 1$), f is defined and we may consider $F_j(w) = \psi \circ f \circ \psi_j^{-1}$ as a germ at infinity defined on H'_δ, $\delta \ll 1$.

Notice that $\boxed{F_j(w) = w + 1 + O(w^{-\frac{1}{r}})}$ $(*_1)$ In fact,

$f(z) = z(1 - z^r + O(z^{r+1}))$, and, setting $z = \psi_j^{-1}(w)$:

$$F_j(w) = \Psi \circ f(z) = \frac{1}{2z^2(1-z^2+O(z^{2\nu+1}))^2} = w\cdot(1+\underset{\frac{1}{w}}{\underset{\shortparallel}{2z^2}}+O(z^{2\nu+1})) =$$

$$= w+1+O(w^{-\frac{1}{2}}).$$

Rem: By the formal classification, we may assume $f(z)=z(1-z^2+O(z^{22}))$, which gives $F_j(w)=w+1+O(\frac{1}{w})$.

Step 2: Construct small petals

In particular, for any $R>>0$, $\exists C=C(R)>>0$ so that $\forall w$: $|w|>R \Rightarrow |F(w)-w-1| \leq \frac{C}{|w|^{\frac{1}{2}}}$ $\left(<\frac{C}{R^{\frac{1}{2}}}\right)$

In particular $\boxed{\forall \varepsilon>0\ \exists C,\ R=R(\varepsilon)}$ (does not depend on ε) (so that $\frac{C}{R^{1/2}}<\varepsilon$) so that: (✱2)

$\boxed{|F(w)-w-1|<\varepsilon.}$ We deduce $\mathrm{Re}\,F(w) > \mathrm{Re}(w) + (1-\varepsilon)$. For $\frac{1}{2\tau\delta}>R$, $F(H_\delta')\subset H_\delta'$ and by recursion $\mathrm{Re}\,F^n(w) \geq \mathrm{Re}\,w + n(1-\varepsilon)$

$\Rightarrow F^n(w)\to\infty$ (and $f^n(z)\to 0$) $\forall w\in H_\delta'$ $(z\in B_{\delta,j})$.

We show that also the argument of $F^n(w)$ tends to 0, (i.e. $f^n(z)\to 0$ tangent to v_j^+). In fact:

$$\boxed{\frac{F^n(w)}{n}} = \frac{F^{n-1}(w)}{n}+1+\frac{1}{n}O((F^{n-1}(w))^{-\frac{1}{2}}) = \dots \boxed{= \frac{w}{n}+1+\frac{1}{n}\sum_{k=0}^{n-1}O((F^k(w))^{-\frac{1}{2}})} \quad (✱_3)$$

Since $F^k(w)\to\infty$, $O(F^k(w)^{-\frac{1}{2}})\to 0$ and by Cesaro summability theorem

$$\frac{1}{n}\sum_{k=0}^{n-1}O((F^k(w))^{-\frac{1}{2}})\to 0, \quad \text{and} \quad \frac{F^n(w)}{n}\to 1.$$

In particular, $\arg F^n(w)\to 0$. $\triangle$

Since $B_{\delta,j}$ is invariant and a neighborhood of the attracting direction v_j^+, it is an attracting petal.

Repeating the same argument for F^{-1}, we can construct petals (repelling) $B^-_{\delta,j}$, but $B^-_{\delta,j}$ and $B^+_{\delta,j}$ would be only tangent and not intersecting.

We need larger petals, which correspond to find F_j-invariant sets $V_{\delta,\varepsilon}=V$ bigger than H'_δ, and so that $\varphi_j^{-1}(V_j)$ for bigger amplitude than $B_{\delta,j}$
P_j^+

Step 3: Construct large petals (1st property)

$\forall \varepsilon>0$,
Fix $C, R=R(\frac{\varepsilon}{2})$ as above ~~and up to taking R bigger, we may assume that~~ so that:

$|F(w)-w-1|<\frac{\varepsilon}{2} \quad \forall |w|>R. \quad (*)$

Pick δ so that $\frac{1}{2\pi\delta}>R$, and set $V:=H'_\delta \cup \{w\in\mathbb{C} \mid |\operatorname{Im} w|>-\varepsilon \operatorname{Re} w+M_\varepsilon\}$,

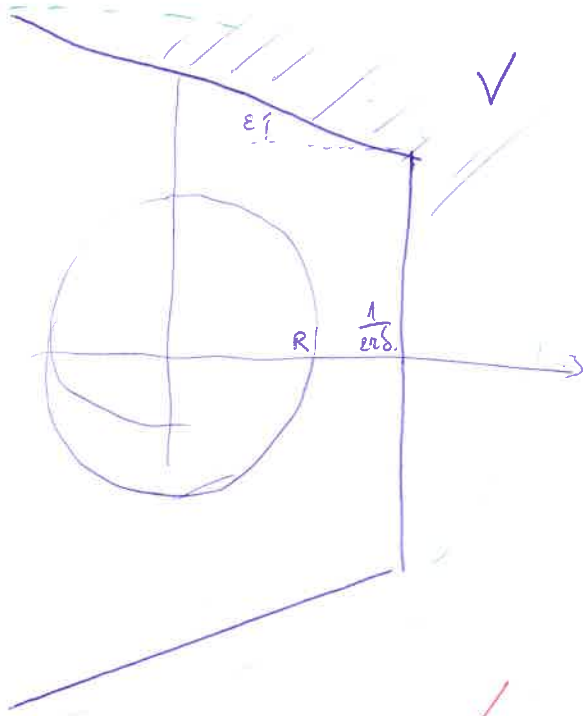
with M_ε big enough so that $V\subset\{w, |w|>R\}$ $\left(M_\varepsilon=\frac{1+\varepsilon}{2\pi\delta}\right)$.

Then from $(*)$ we infer: $\operatorname{Re}(F(w))>\operatorname{Re} w+1-\frac{\varepsilon}{2}$; $|\operatorname{Im} F(w)-\operatorname{Im} w|<\frac{\varepsilon}{2}$
$\forall w\in V$

It is easy to check that V is invariant.

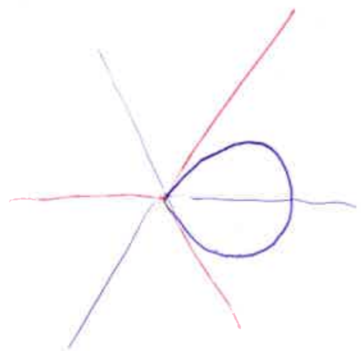
$|\operatorname{Im} F(w)| \geq |\operatorname{Im} w|-\frac{\varepsilon}{2} > -\frac{\varepsilon}{2}-\varepsilon \operatorname{Re} w+M_\varepsilon$

$> -\varepsilon \operatorname{Re} F(w)+M_\varepsilon+\underbrace{\frac{\varepsilon}{2}-\frac{\varepsilon^2}{2}}_{\geq 0}$



Moreover again $\operatorname{Re} F^n(w)>\operatorname{Re} w+\frac{n}{2}$, and every orbit of V eventually enters H'_δ.

Hence $\varphi_j^{-1}(V)=:P_j^+$ is again an attracting petal, but this time containing ample sector.



Rem: taking $\bigcup_\varepsilon V_\varepsilon=V$, we could find invariant petals whose boundary is tangent to the repelling directions.
V depends on ε (denote V_ε)

Notice that $\operatorname{Re} F^n(w) \to +\infty$ $\forall w \in V$.

In particular, $\forall z \in P_j^+$, $\exists n_0 \in \mathbb{N}$ so that if $f^n(z_{-n}) = z$, then $z_{-n} \notin P_j^+$.

Apply this to f^{-1} and repelling petals, and we get that orbits of points (not in 0) eventually leave repelling petals.

This implies, by computing $K_f(0) = \bigcap_{n\geq 0} f^n(U)$, U: union of attracting and repelling petals, that $\forall z \in K_f$ $f^n(z)$ belongs to an attracting petal for $n >> 0$.

We now prove that $F(w) = w + 1 + \mathcal{O}(w^{-\frac{1}{2}})$ is holomorphically conjugated to the translation $w \mapsto w+1$.

Step 4: $F \sim w \mapsto w+1$.

Claim 1: $\boxed{\left|F'(w) - 1\right| \leq \frac{C}{S^{1+\frac{1}{2}}}}$ $\forall w$, $|w| \geq 2S \geq 2R$.

In fact, $F(w) - w - 1$ sends ~~the~~ a disc of center w and radius S to a disc of center 0 and radius $\frac{C}{S^{\frac{1}{2}}}$, as far as $|w| - S > R$.

By the Cauchy estimate we obtain the estimate for the derivative at w. $\triangle$

Now, for any $w', w'' \in \{\operatorname{Re} w \geq 2S\} \subset H_S$ (H_{2S}). By Lagrange applied to $F(w) - w$, we get,

we get $\left|\frac{F(w'') - F(w')}{w'' - w'} - 1\right| \leq \left|F'(w) - 1\right| \leq \frac{C}{S^{1+\frac{1}{2}}}$, $\exists w \in [w', w'']$.

Pick some base point $\hat{w}$, $\operatorname{Re} \hat{w} > 2R$, and set:

$\beta_k(w) = F^k(w) - F^k(\hat{w})$.

$\left|\frac{\beta_{k+1}(w)}{\beta_k(w)} - 1\right| \leq \frac{C'}{k^{1+\frac{1}{2}}}$ since $F^k(w), F^k(\hat{w}_0) \in \{\operatorname{Re} w > \frac{k}{2}\}$

where $C' = C \cdot 4^{1+\frac{1}{2}}$

Since $\sum \frac{1}{k^{1+\frac{1}{2}}} < +\infty$, the product $\prod_{k=1}^{\infty} \frac{\beta_{k+1}(w)}{\beta_k(w)}$ converges uniformly on compacts to a holomorphic map, and so does the sequence $\beta_n(w)$.

towards some function $\beta_\infty(w)$, holomorphic on $\{\mathrm{Re}\, w > 2R\}$.

Since ~~the~~ $F^{\wedge}$ is injective, so is β_n, and β_∞ as the limit by Hurwitz Thm. (β_∞ is either injective or constant)

(In fact, β_∞ is not constant since it satisfies:)

$$\beta_k \circ F(w) = F^{k+1}(w) - F^k(\hat{w}) = \beta_{k+1}(w) + \underbrace{F^{k+1}(\hat{w}) - F^k(\hat{w})}_{\downarrow \; 1}$$

$$\searrow \beta_\infty \circ F(w) = \overset{\downarrow}{\beta_\infty}(w) + 1 \quad \text{(Abel equation).}$$

by (*2) & $F^k(\hat{w}) \to \infty$.

$$\boxed{\text{Claim: } \lim_{\substack{w\to\infty \\ w \in H_S'}} \frac{\beta_\infty(w)}{w} = 1.}$$

In fact, $\forall \eta > 0$, $\exists k_0$ s.t. $\left| \dfrac{\beta_\infty(w) - \beta_{k_0}(w)}{w - \hat{w}_0} \right| < \dfrac{\eta}{3}$. (*A)

(uniform convergence of $\dfrac{\beta_k}{\beta_0}$ to $\dfrac{\beta_\infty}{\beta_0}$)

Moreover, $\left| \dfrac{\beta_{k_0}(w)}{w - \hat{w}} - 1 \right| = \left| \dfrac{F^{k_0}(w) - F^{k_0}(\hat{w}) + \hat{w} - w}{w - \hat{w}} \right| =$

$$\overset{(*)}{=} \left| \frac{\cancel{w} + k_0 + \sum_{k=0}^{k_0-1} O\big((F^k(w))^{-\frac{1}{2}}\big)^{\to 0} - \cancel{w} + \hat{w} - F^{k_0}(\hat{w})}{w - \hat{w}} \right| = O\left(\frac{1}{|w|}\right)$$

In particular $\exists R >> 0$ so that $\forall w$, $|w| > R$, we get $\left| \dfrac{\beta_{k_0}(w)}{w - \hat{w}} - 1 \right| < \dfrac{\eta}{3}$. (*B)

Finally, $\left| \dfrac{\beta_\infty(w)}{w - \hat{w}} - \dfrac{\beta_\infty(w)}{w} \right| = \left| \dfrac{\beta_\infty(w)}{w - \hat{w}} \right| \cdot \left| \dfrac{\hat{w}}{w} \right| < \dfrac{\eta}{3}$ for $|w| >> 0$.

(*C)

In fact, $\left|\dfrac{\hat{w}}{w}\right| \to 0$, $\left| \dfrac{\beta_\infty(w)}{w - \hat{w}} \right| \sim \left| \dfrac{\beta_{k_0}(w)}{w - \hat{w}} \right| \sim 1$.

(*)B

$$\text{Hence } \left| \frac{\beta_\infty(w)}{w} - 1 \right| \le \underbrace{\left| \frac{\beta_\infty(w)}{w} - \frac{\beta_\infty(w)}{w - \hat{w}_0} \right|}_{\wedge\, \frac{\eta}{3} \;(*C)} + \underbrace{\left| \frac{\beta_\infty(w)}{w - \hat{w}} - \frac{\beta_{k_0}(w)}{w - \hat{w}} \right|}_{\wedge\, \frac{\eta}{3} \;(*A)} + \underbrace{\left| \frac{\beta_{k_0}(w)}{w - \hat{w}} - 1 \right|}_{\wedge\, \frac{\eta}{3} \;(*B)} < \eta.$$

It remains to prove that β_∞ sends ~~a~~ H_S' to some open set containing H_R for $R >> 0$.

Since $\frac{\beta_\infty(w)}{w} \to 1$ for $w \in H_R$, $w \to \infty$, there exists $S >> 0$ so that

$|\beta_\infty(w) - w| < \frac{|w|}{3}$ for $|w| > S$, $w \in H_R$ ($\Leftarrow w \in H_S$)

~~Th~~ Let w_0 be any point in H_{2S}, so that $\overline{D} = \overline{D(w_0, \frac{|w_0|}{2})} \subset H_S$.

Then $\forall w \in \overline{D}$, we get that $S < |w| < \frac{3}{2}|w_0|$, and $|\beta_\infty(w) - w| < \frac{|w_0|}{2}$.

By Rouché's theorem, $\beta_\infty(w) - w_0$ and $w - w_0$ have the same number of zeroes in D (i.e. 1), and $w_0 \in \operatorname{Im} \beta_\infty|_{H_R}$

$\boxed{\text{This argument also proves injectivity. Because for } |w - w_0| > \frac{|w_0|}{2},\ |F(w) - w| < 1 + \frac{\varepsilon}{2} < \frac{|w_0|}{2} \text{ for } |w_0| >> 1}$ Δ

We can conclude the proof of the theorem: (part of $\exists!$ solution to Abel functional eq.)

($\exists$) We set $\phi(z) = \beta_\infty \circ \varphi(z)$ for $z \in B_\delta$, $\delta >> 0$.

To define ϕ on the whole petal P: $\forall z \in P$ $\exists k = k(z)$ so that $f^k(z) \in B_\delta$.

We set $\phi(z) := \phi(f^k(z)) - k$.

It is easy to check that this definition does not depend on the choice of the (sufficiently large) k, and again satisfies the Abel equation.

It is also injective, since $\phi(z_1) = \phi(z_2) \Leftrightarrow \phi(f^k(z_1)) - k = \phi(f^k(z_2)) - k \Leftrightarrow f^k(z_1) = f^k(z_2) \Leftrightarrow z_1 = z_2$ being f an automorphism

(!) Since any Fatou coordinate ϕ on P can be restricted to $B_S \cap P$ and then extended uniquely on B_S (as above), we may assume $P = B_S$.

Let ϕ_1, ϕ_2 be two Fatou coordinates, then $g = \phi_2 \circ \phi_1^{-1}$ satisfies

$g(w+1) = g(w) + 1$ ($g : U_1 \to U_2$ bijection, U_1, U_2 containing H_R for $R >> 0$).

We can extend g to a map $G : \mathbb{C} \to \mathbb{C}$ by setting $G(w) = g(w+k) - k$ for $k >> 0$.

Since $G:\mathbb{C}\to\mathbb{C}$ is an automorphism, it is affine: $g(z)=az+b$.
From the functional equation we get $g(1)=a+b=g(0)+1=b+1 \Rightarrow a=1$
and g is a translation. $\square$

Corollary: For each attracting or repelling petal P, the quotient manifold $P/\langle f\rangle$ is conformally isomorphic to the infinite cylinder $\mathbb{C}/\mathbb{Z}$.

Proof: $P/\langle f\rangle$ does not depend on P, but only on the attracting/repelling direction v associated. In fact:
If P_1 and P_2 are petals associated to v, then $P=P_1\cap P_2$ also is, and
$i_j: P \hookrightarrow P_j$ induce isomorphisms $I_j: P/\langle f\rangle \to P_j/\langle f\rangle$.

In particular, one can consider $P=B_\delta$ ~~…~~ and its image through ϕ the Fatou coordinate $\rightsquigarrow$ $B_\delta/\langle f\rangle \simeq \mathbb{C}/\langle z\mapsto z+1\rangle = \mathbb{C}/\mathbb{Z}$. $\square$

$P/\langle f\rangle$ is called "Écalle cylinder".
The behavior of such cylinders / Fatou coordinates under perturbation is quite important (in Bifurcation theory).

Corollary: (speed of convergence)
Let $f \in \mathbb{C}\{z\}$ be a tangent to the identity germ of multiplicity $r+1$.
Then $\forall z$ so that $f^n(z)\to 0$, $|f^n(z)| \propto n^{-\frac{1}{r}}$: i.e. $\lim_{n\to\infty} \frac{|f^n(z)|}{\frac{1}{\sqrt[r]{n}}} \in (0,+\infty)$

Proof: direct computation from: $\varphi(z)=\frac{1}{rz^r}$, $\phi \sim \frac{1}{rz^r}$, $f^n \simeq z\mapsto z+n$.